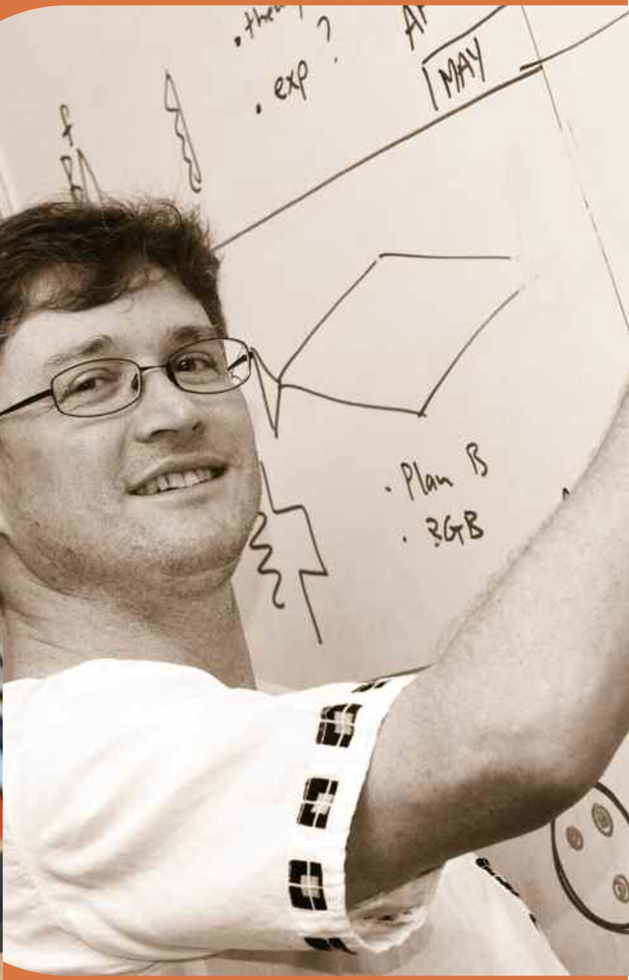
Mathematical modelling

A MATHEMATICAL MODEL involves an abstract structure that uses a mathematical language to describe a system. This abstract structure sometimes depends on the preference and expertise of the modeller and includes, but is not limited to, dynamical systems, statistical models or differential equations. The process of mathematical modelling usually begins with the desire to describe a situation in the real world. In mathematical modelling the goal is to express the real-world problem in symbolic terms that lead to a set of variables and a set of equations that establishes the relationship between the variables. The simple structure allows the modeller to gain insight and clarity about the system because the model is usually designed to describe some aspects accurately, while omitting less relevant details. **– Dr Rebecca Maserumule**

Dr Rebecca Maserumule's research interests involve the use of partial differential equations and numerical computing with and emphasis on inverse modelling in geoscience.

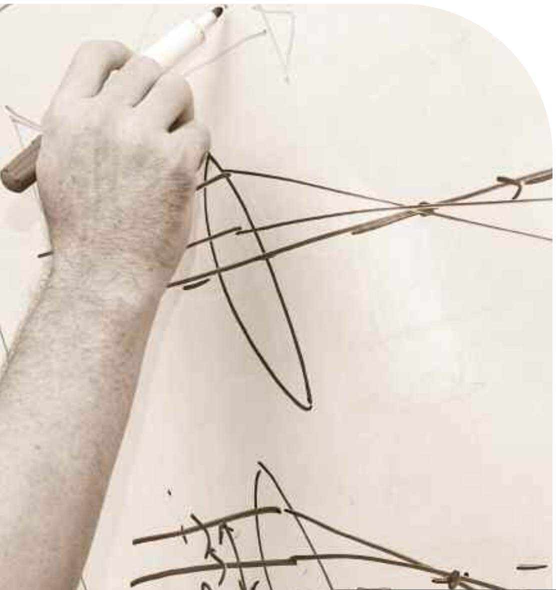


Porro-prism laser enigma

unravelled through mathematical optics

by Dr Andrew Forbes

AN ENIGMA FOR 30 YEARS, a problem in Porro-prism lasers has now been solved by CSIR researchers by applying basic arguments of mathematical symmetry, together with new physical insight into intra-cavity Porro prisms.



The mathematical optics team at the CSIR National Laser Centre concentrates on solving problems in the realms of optics and lasers by approaching problems with mathematical rigour, while recognising the importance of experimental verification. Topics currently studied include creating novel laser beams for biological trapping and tweezing (the technique of using a focused laser beam as an atom trap), propagation of laser beams through a turbulent atmosphere to improve telecommunications signals and creating new laser systems based on micro-optical elements.

This article outlines the approach taken in the recent success of the team, comprising research group leader Dr Andrew Forbes, Igor Litvin (PhD student) and Liesl Burger (MSc student and staff member), and illustrates the power of coupling mathematical, physical and computational tools to solve seemingly intractable problems.

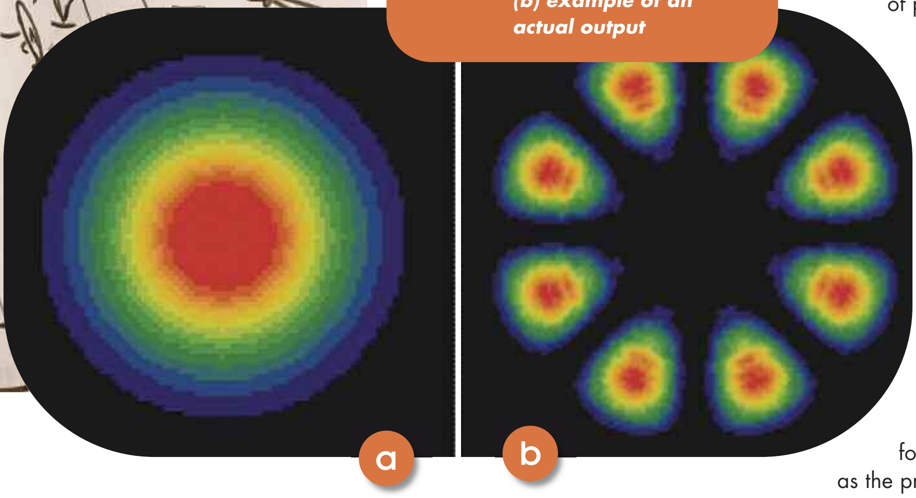


Figure 1 (a) Example of an expected laser output (Gaussian beam) (b) example of an actual output

Getting to the bottom of 'temperamental' Porro-prism lasers

Porro prisms (right angle prisms) have the property that all incident rays on the prism are reflected back parallel to the initial propagation direction, independent of the angle of incidence, thus making these insensitive to misalignment. In a Porro-prism laser the traditional end mirrors are replaced by Porro prisms. Such lasers have been exploited for their ruggedness, and used where a laser beam is required at a large distance from the source and where the source is not a stable platform; for example, range finding and laser designators. Despite the ubiquitous nature of these lasers, resonators with internal Porro prisms have not been well understood for nearly 30 years. One would expect an output beam similar to that shown in Figure 1(a), whereas in some lasers the observed output beam was radically different, as illustrated in Figure 1(b), exhibiting petal-like patterns of varying numbers of petals. There seemed no logical reason for these petals, and more bizarrely, the number of petals would vary from laser to laser. Sometimes these lasers would even not work at all, for no apparent reason! Unsurprisingly, the answer was in how one viewed the prisms.

In early attempts to model such lasers, the prisms were treated as though they were perfect mirrors – any incoming ray is returned out, except for an inversion about the reflecting point. This approach was followed and adopted over time as the preferred model for intra-cavity prisms. Since it wasn't working, the question begged: If prisms could not be treated like mirrors, how should they be described?

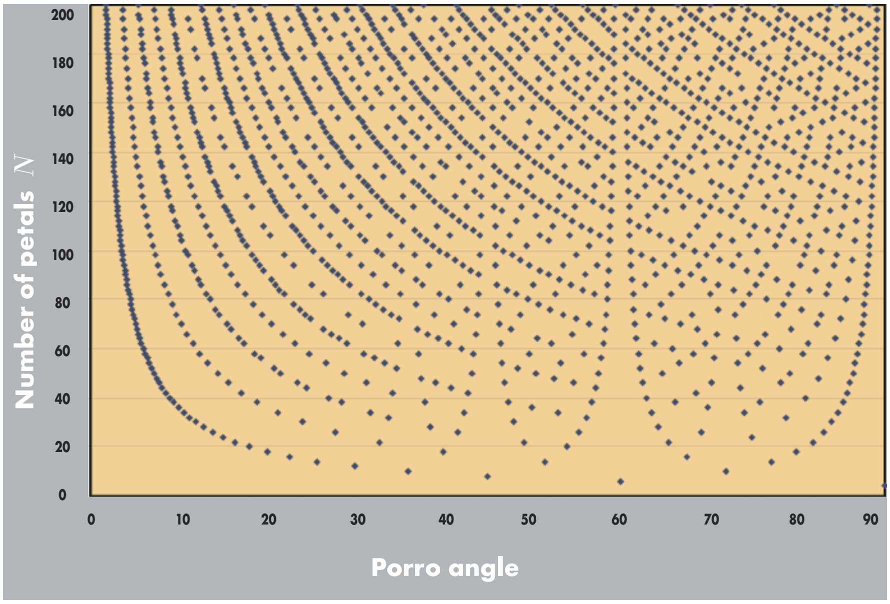


Figure 2 ***Plot of the discrete set of angles that gives rise to a petal pattern, with the corresponding number of petals to be observed. Data calculated for $m \in [1,100]$ and $i \in [1,50]$***

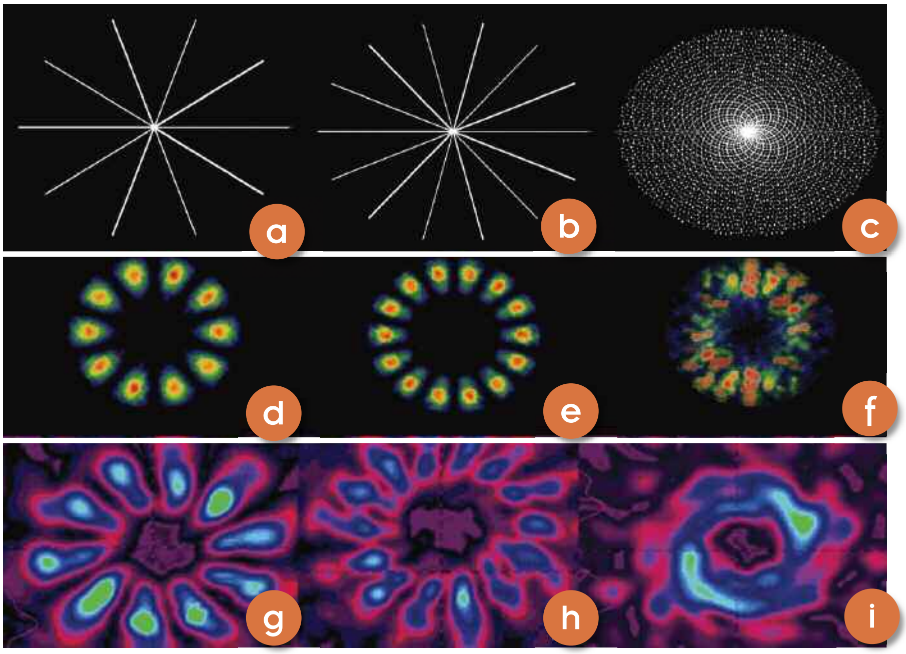


Figure 3 ***The analytical model depiction of finitely sub-divided fields in (a) and (b), and an infinitely sub-divided field in (c). Numerically this results in a pattern with (d) 10 petals, (e) 14 petals and (f) no petals. The corresponding experimentally-observed output is shown in (g) – (i)***

Mathematics of symmetries

The researchers approached the problem from a fundamental basis, realising that without a model for the prism, the laser could never be described completely. Two new concepts were introduced – firstly, the prism edges would lead to high diffraction losses and would appear as linear loss lines across the resonating optical field.

Secondly, these linear loss lines would appear to the optical field inside the resonator to be rotating in space, due to the inversion properties of the prisms. To convert these statements into a mathematical expression required consideration of the mathematics of symmetries.

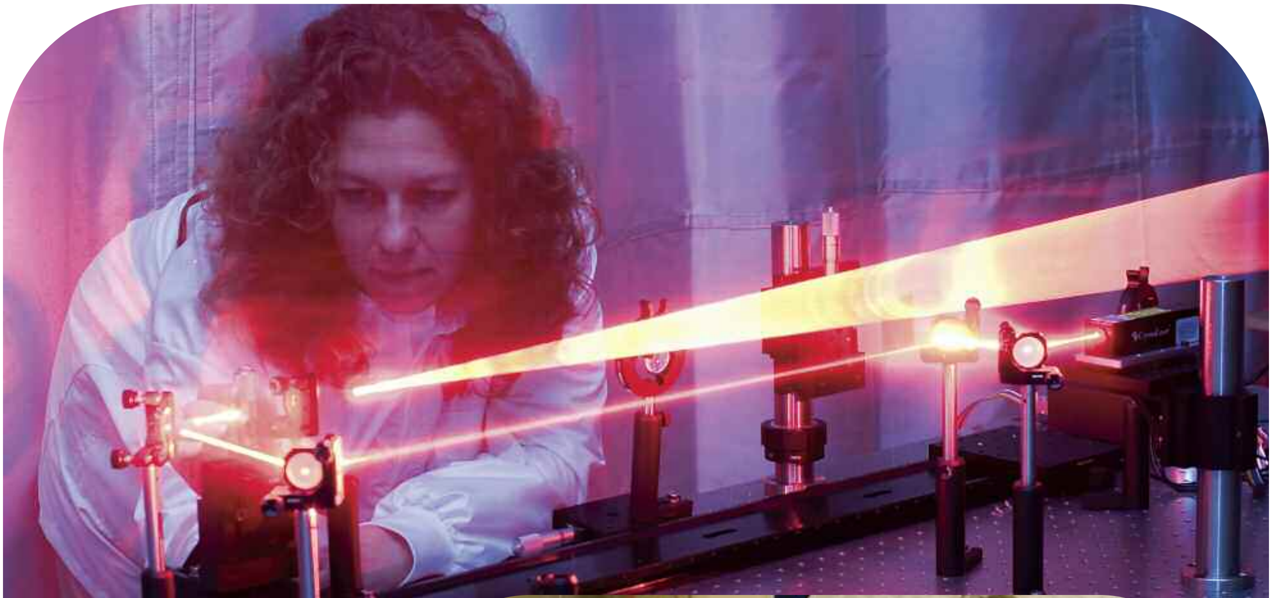
The initial positions of the two prism edges act as mirror image planes – they result in symmetry about each plane after each reflection. These two symmetry planes result in very complex inversions of the loss lines after each pass in the resonator, with the loss lines appearing to rotate in space. Laser scientists asked the question: Will these loss lines ever cycle into a repeating pattern? They were able to derive a simple expression for very specific angles at which this would happen:

$$\alpha = \frac{i\pi}{m} \qquad (1)$$

for positive integers i and m. From this they could also derive the number of subdivisions or petals to be expected:

$$N = \frac{j\,2\pi}{\alpha} \qquad (2)$$

for certain integers j. The implication of this is startling:



Two of the CSIR scientists who refused to be baffled by the puzzling, petal-like patterns of porro-prism lasers, Dr Andrew Forbes and Liesl Burger

Only at very specific angles will the field be finitely sub-divided, thus leading to some regions with low loss for lasing to take place. In addition, since the position of these sub-divisions remains stable after a certain number of round trips, the modal pattern that oscillates inside such a resonator will give rise to a petal-like pattern. At other Porro angles the high loss apexes will continuously rotate to new positions, thus resulting in high losses across the entire field. For example, at 30° a laser beam with 18 petals is predicted, at 36° a laser beam with 10 petals is expected, while at 30,5° no stable laser beam is expected. The complete set of calculated stable angles with associated number of petals is shown in Figure 2.

From mathematics to experiment

With this new mathematical model the researchers were able to put together a physical optics computational model of the laser resonator to determine the output laser beam. Having a prediction of the laser beam is crucial for experimental verification of the model. The complete transition from mathematics to experiment is shown in Figure 3. Physical insight, expressed in the language of mathematics and enhanced with computational techniques, allowed a new model to be created of Porro-prism lasers that correctly predicts the observed output. It can now, for the first time, be explained why the so-called petal patterns sometimes seen from such lasers exist, say when these will exist, and predict how many petals will be seen. This new mathematical model also explains why such lasers sometimes do not work, even though very little would appear to have changed in the laser – perhaps only a 0,5° offset in the prism orientations – and opens the way to the restraints on future designs of Porro-prism lasers.

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References and further information

IA Litvin, L Burger and A Forbes, 'Petal-like modes in Porro-prism resonators', *Opt Express* 15, 14 065–14 077 (2007).

Movies and further information: www.csir.co.za/lasers/mathematical_optics.html